

# Chaos, Order Statistics and Unstable Periodic Orbits

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We present a new method for locating unstable periodic points of one dimensional chaotic maps. This method is based on order statistics. The densities of various maxima of the iterates are discontinuous exactly at unstable periodic points of the map. This is illustrated using logistic map where densities corresponding to a small number of iterates have been obtained in closed form. This technique can be applied to the class of continuous time systems where the successive maxima of the time series behave as if they were generated from a unimodal map. This is demonstrated using Lorenz model.

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Existence of a dense set of unstable periodic orbits (UPO) is one of the characteristic properties of a chaotic system [1]. These orbits represent the skeleton for the strange attractor of dissipative dynamical systems. Many quantities that characterize chaos in the system, such as the fractal dimension, average Lyapunov exponent, entropy and the invariant measure of the corresponding attractor can be determined by knowing the properties of UPO [2,3,4,5,6]. Extraction of UPO is a necessary step in several studies. For example, knowledge of the locations of various cycles is necessary for control of chaos [7]. Cycles are found to be useful in the synchronization of chaotic signals [8]. Moreover, the quantum mechanical properties of classically chaotic conservative systems have, in the semiclassical regime, a series expansion with respect to the lengths and stability coefficients of the periodic orbits [9]. The importance of unstable periodic orbits and their detection has attracted the attention of several researchers and a number of numerical methods have since been developed to extract unstable periodic orbits [2,10,11,12,13,14,15].

In this letter we present a new technique to extract the UPO of one dimensional chaotic maps using order statistics. This method works for all one dimensional chaotic maps for which the existence of invariant density is guaranteed.

The theory of extreme values and its generalization to order statistics is a classic subject and is extensively used in the study of independent and identically distributed random variables [16]. Let  $\{X_1, X_2, \dots, X_n\}$  be an  $n$ -point set and  $M_n^k$  ( $1 \leq k \leq n$ ) be its  $k^{th}$  maximum i.e.,  $M_n^n \leq M_n^{n-1} \leq \dots \leq M_n^k \leq \dots \leq M_n^2 \leq M_n^1$ . Order statistics is the study of distributional properties of  $M_n^k$ . In the context one dimensional chaotic map, it has been shown that the extreme value density is discontinuous on a set of points belonging to its unstable periodic orbits [17]. However, this method cannot locate all the periodic

points. This has motivated us to investigate whether all the unstable periodic points can be extracted by using order statistics instead of simple extreme value analysis.

Let  $f(x) : [a, b] \rightarrow [a, b]$  be a continuous, one dimensional chaotic map with an invariant density  $\rho_s(x)$ . Let  $\{x_0, x_1 = f(x_0), \dots, x_{n-1} = f^{n-1}(x_0)\}$  be an  $n$ -point set. The density of the  $k^{th}$  maximum is denoted by  $\rho_n^k(x)$ .

Let  $S_n^k$  be the set of locations of the discontinuities of  $\rho_n^k$ . We observe that  $O_n = \cup_{k=1}^n S_n^k$  for all  $n$ , where  $O_n$  is the set of interior (excluding the end points a,b) periodic orbits of all orders strictly less than  $n$ . An outline of the proof of the above result is given towards the end, see for details [18]. This remarkable correspondence between the densities of the order statistics and UPO enables one to extract the periodic points of any order by choosing an appropriate  $n$ . The locations of the cycles are, thus, read off from the extremely sharp discontinuities of the densities.

We now illustrate the method by applying it to logistic map,  $x_{n+1} = f(x_n) : [0, 1] \rightarrow [0, 1] = \lambda x_n(1 - x_n)$ , for  $\lambda = 4$ , a well studied unimodal map exhibiting chaos [1].

We analytically derive the expression for the densities of the first and second maxima of a three point set of logistic map using the formulation described in equation (5). They are

$$\rho_3^1(x) = \begin{cases} \frac{\sin^{-1}\sqrt{x}}{2\pi} & 0 \leq x < \frac{3}{4} \\ \frac{7\sin^{-1}\sqrt{x}}{2\pi} - 1 & \frac{3}{4} \leq x < \frac{5+\sqrt{5}}{8} \\ \frac{6\sin^{-1}\sqrt{x}}{\pi} - 2 & \frac{5+\sqrt{5}}{8} \leq x < 1 \end{cases} \quad (1)$$

$$\rho_3^2(x) = \begin{cases} \frac{3\sin^{-1}\sqrt{x}}{2\pi} & 0 \leq x < \frac{5-\sqrt{5}}{8} \\ \frac{4\sin^{-1}\sqrt{x}}{2\pi} - \frac{1}{2} & \frac{5-\sqrt{5}}{8} \leq x < \frac{3}{4} \\ \frac{5\sin^{-1}\sqrt{x}}{2\pi} & \frac{3}{4} \leq x < \frac{5+\sqrt{5}}{8} \\ 1 & \frac{5+\sqrt{5}}{8} \leq x < 1 \end{cases} \quad (2)$$

Note that the point  $x = (5 - \sqrt{5})/8$  is not present in  $S_3^1 = \{3/4, (5 + \sqrt{5})/8\}$ , but is a periodic point of order two, which clearly shows that extreme value statistics is not adequate to extract all the UPOs and one needs order statistics.

Numerically, given a map with an invariant density (a mere existence is necessary and sufficient to render meaning to the averaging procedure), one computes  $\rho_n^k$  in the following way. Starting with an initial condition,  $n - 1$  successive iterates of the map are obtained and arranged in the ascending order of their magnitude. The  $k^{th}$  maximum is thus picked up. This is repeated for many initial conditions to obtain a histogram representing  $\rho_n^k$ . For the logistic map  $\rho_4^k(x)$  has been computed numerically and is shown in fig 1.

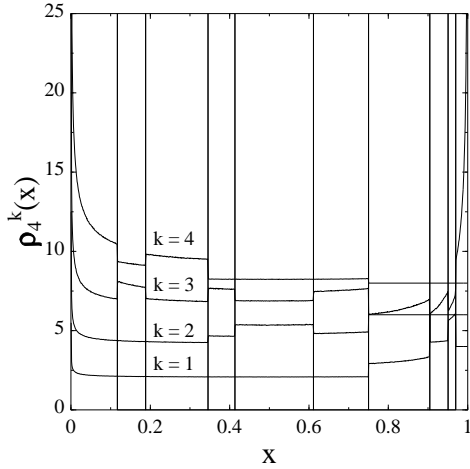


FIG. 1. Numerically computed  $\rho_4^k$  for  $k = 1, 2, 3$  and  $4$  of the logistic map ( $10^7$  initial conditions, bin width=0.001). The locations of the discontinuities and the cycle points of all orders  $< 4$  are compared. The cycle points are indicated by vertical lines. Note the absence of discontinuity at the origin which is a fixed point (see the text).

Further, the topological entropy, which is a quantitative characterization of chaos in the system, can be estimated [19] from the number of periodic orbits as follows,

$$h = \lim_{n \rightarrow \infty} \frac{1}{n} \ln N(n) \quad (3)$$

where  $N(n)$  is the total number of periodic points of order  $n$ . We denote the number of elements in a set  $A$  by  $\text{Card}(A)$ . Since  $N(n) = \text{Card}(O_{n+1}) - \text{Card}(O_n)$ , it can be obtained from the discontinuities of  $\rho_{n+1}^k(x)$  ( $k = 1 \cdots n + 1$ ) and that of  $\rho_n^k(x)$  ( $k = 1 \cdots n$ ). In the case of logistic map we obtain  $\text{Card}(O_n)$  as 3, 9, 21, 51 and 105 for  $n = 3, 4, 5, 6$  and  $7$  respectively. Since this formalism is based on the discontinuities of the density over an interval  $[a, b]$ , it cannot indicate a periodic point occurring at the extreme points of the interval  $a$  and  $b$ . Thus, in

the above list (see also fig 1), the fixed point at the origin is not included. The topological entropy calculated is 0.6648, which is within 4% of the exact value  $\ln 2$ .

However, for  $\lambda < 4$ , the invariant density,  $\rho_s(x)$  is not available in closed form. The existence of  $\rho_s(x)$  can be checked and is sufficient for computing order densities. For most  $\lambda < 4$ ,  $\rho_s(x)$  itself is discontinuous on a large number of points. Thus, the corresponding order densities pick up discontinuities at those points at which  $\rho_s(x)$  is discontinuous and also at unstable periodic points. Since the strength of the inherent discontinuities of  $\rho_s(x)$  in  $\rho_n^k(x)$  is large compared to that of the discontinuities at periodic points, it is numerically difficult to count or detect their locations.

There are situations where the essence of the dynamics of a continuous time dynamical system is captured effectively by one dimensional maps or their equivalent. In his classic paper Lorenz showed that the successive peaks of a one dimensional time series behave like iterates of a map [20], see also [21]. We work with the Lorenz model [20]

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(r - z) - y, \quad \dot{z} = xy - bz \quad (4)$$

for parameters  $r = 28$ ,  $\sigma = 10$  and  $b = 8/3$ .

We compute the  $\rho_n^k(x)$  for different  $n$  for the map constructed out of the successive maxima of the time series corresponding to the state space variable  $z$ . Existence of an invariant density for the so constructed map is verified numerically.  $\rho_n^k(x)$  of this map also is discontinuous on a set of points. The topological entropy of the map is calculated using equation (3). For  $n=3, 4, 5$  and  $6$ , the topological entropy  $h=0.6486, 0.7361, 0.7700$  and  $0.7708$  respectively, showing a reasonable convergence in the numerical value of topological entropy.

It is important to consider the effect of noise on the order densities to explore the possible applicability of this method to an experimental time series convoluted with noise. Here, the distribution and strength of the noise play a decisive role. Our preliminary investigations involve addition of noise, generated from uniform distribution, to the iterates of the logistic map with  $\lambda = 4$ . The strength of the noise is 4% of the variance of the iterates. The spill over from  $[0, 1]$  is reinjected. The strength of the discontinuity at the periodic points is observed to diminish with increase in the strength of the noise.

We now turn to establish the contribution of periodic points to the order statistics rigorously. The formulation of the order statistics involves the probability  $P_n^k(x)$  that  $M_n^k \in (x, x + dx)$  and is given by

$$P_n^k(x) = \frac{1}{(k-1)!} \sum_{i_1=0}^{n-1} \sum_{\mathcal{I}} \text{Prob} \left( \begin{array}{c} x \leq x_{i_1} \leq x + dx \\ x_i > x | i \in \mathcal{I} \\ x_j < x | j \in \mathcal{J} \end{array} \right) \quad (5)$$

where  $\mathcal{I} = \{i_2, \dots, i_k\}$  and  $\mathcal{J} = \{i_{k+1}, \dots, i_n\}$  with  $\{i_1, \dots, i_n\}$  being a permutation of  $\{0, \dots, n-1\}$ . Using the joint probability density of the  $n$  points together with the above equation, we derive an expression for the cumulative distribution  $F_n^k(x)$  of the  $k^{th}$  maximum. The order density  $\rho_n^k(x)$  is the derivative of  $F_n^k(x)$ .

Our central result is : given a continuous one dimensional map  $f : [a, b] \rightarrow [a, b]$  with a continuous invariant density  $\rho_s(x)$ , the order density  $\rho_n^k(x)$  is discontinuous at an interior point  $x_*$  if and only if  $x_*$  is an unstable periodic point such that  $f^l(x_*) = x_*$  with  $l \leq n-1$ .

We give an outline of the proof and the notation used is described here. (i) A periodic point of order  $l$  is denoted by  $p_{jl\beta}$ , i.e.,  $f^l(p_{jl\beta}) = p_{jl\beta}$ . The index  $\beta = 1 \dots l$  corresponds to the distinct points of the  $l$ -cycle. In a chaotic map, in general, there exist more than one  $l$ -cycles, and the number of  $l$ -cycles  $N_l$  increases with  $l$ . The index  $j$  runs over the different  $l$ -cycles and hence  $j = 1, \dots, N_l$ . For example, in logistic map, we have two orbits of period 3. The periodic points of a given  $l$ -cycle are ordered such that  $p_{jll} = \max_{\beta} \{p_{jl\beta}\}$  corresponds to the maximum of that cycle. (ii) Let  $y = f^i(x)$ . The set of preimages of  $x$  with respect to  $f^i$  are  $\{g_{i\alpha}(x) | f^i(g_{i\alpha}(x)) = x\}$ ,  $\alpha = 1, \dots, I_i$  where  $I_i$  is the number of preimages. For example, the preimages of  $x = 3/4$  of the logistic map are  $\{3/4, 1/4\}$ .

As a first step, we show that the extreme value density,  $\rho_n^1(x)$  is discontinuous at  $x$  if and only if  $x$  is the maximum point of a periodic orbit, i.e.,  $x = p_{jll}$ ,  $l \leq n-1, j = 1, \dots, N_l$ . Proving this involves two stages namely, to show that the extreme value density cannot be discontinuous at  $x$  unless  $x$  is the maximum of a periodic orbit and then show that  $\rho_n^1(x)$  is discontinuous at the maximum of all the periodic points. The extreme value density can be formulated by using equation (5) to be

$$\rho_n^1(x) = \rho_s(x) \prod_{j=1}^{n-1} \Theta(x - f^j(x)) + \sum_{i=1}^{n-1} \sum_{\alpha=1}^{I_i} A_{i\alpha}(x) \Theta(x - g_{i\alpha}(x)) \prod_{j \neq i} \Theta(x - f^j(g_{i\alpha}(x))) \quad (6)$$

where  $A_{i\alpha}(x) = \rho_s(g_{i\alpha}(x)) / \left| \frac{d}{dx} f^i(x) \right|_{x=g_{i\alpha}(x)}$ . The above equation can be recast as

$$\rho_n^1(x) = \rho_s(x) C_0(x) + \sum_{i=1}^{n-1} \sum_{\alpha=1}^{I_i} A_{i\alpha}(x) C_{i\alpha}(x) \quad (7)$$

It can be shown that if  $\rho_s(x)$  is continuous then  $A_{i\alpha}(x)$  is also continuous, see [18] for details. As all the terms in equation (6) are positive definite, proving either  $C_0(x)$  or  $C_{i\alpha}(x)$  is discontinuous at a point is sufficient to prove  $\rho_n^1(x)$  to be discontinuous. Also note  $\Theta(z) = (1 + \text{sgn}(z))/2$  is discontinuous only at  $z = 0$ .

Since  $\rho_s(x)$  and hence  $A_{i\alpha}(x)$  are continuous,  $\rho_n^1(x)$  is discontinuous only if either  $\Theta(x - f^l(x))$  for some  $l \leq$

$n-1$ , or  $\Theta(x - g_{i\alpha}(x))$ , or  $\Theta(x - f^m(g_{i\alpha}(x)))$  for some  $m \neq i, m \leq n-1$  is discontinuous.

If  $\Theta(x - f^l(x))$  is discontinuous then  $x = f^l(x)$  implying  $x \in \{p_{jl\beta}\}$ . Considering  $C_0(p_{jl\beta} + \epsilon)$  (where  $\epsilon$  is an infinitesimal quantity of appropriate sign), we have

$$C_0(p_{jl\beta} + \epsilon) = \prod_{k=1}^{n-1} \Theta(p_{jl\beta} - f^k(p_{jl\beta}) + O(\epsilon)) \quad (8)$$

If  $p_{jl\beta} \neq p_{jll}$ , in the product, there exists a  $k_l$  such that  $f^{k_l}(p_{jl\beta}) = p_{jll}$  and the product vanishes as  $p_{jll} > p_{jl\beta}$  for all  $\beta < l$ . This shows that  $C_0(x)$  cannot be discontinuous unless  $x = p_{jll}$ .

$\Theta(x - g_{i\alpha}(x))$  is discontinuous at  $x = g_{i\alpha}(x)$ . We have  $f^i(x) = f^i(g_{i\alpha}(x)) = x$  and this implies  $x \in \{p_{ji\beta}\}$ . Considering  $C_{i\alpha}(p_{ji\beta} + \epsilon)$ , the leading order term is

$$C_{i\alpha}(p_{ji\beta} + \epsilon) = \Theta\left(\epsilon \left[1 - \frac{d}{dx} g_{i\alpha}(x) \Big|_{x=p_{ji\beta}}\right]\right) \prod_{k \neq i} \Theta(p_{ji\beta} - f^k(p_{ji\beta}) + O(\epsilon)) \quad (9)$$

It can be shown that  $g'_{i\alpha}(x) = 1/f'(g_{i\alpha}(x))$  and  $|f'(g_{i\alpha}(p_{ji\beta}))|$  is always greater than one for all the unstable periodic orbits. This means the first  $\Theta$  function in the above equation is always non zero. Thus, by a similar reasoning outlined above, the second term in the product cannot survive unless  $x = p_{jii}$ .

Consider  $\Theta(x - f^m(g_{i\alpha}(x)))$  which is discontinuous at  $x = f^m(g_{i\alpha}(x))$  for some  $m \in \{1, \dots, n-1\} \setminus \{i\}$ . There exists an  $l$  such that  $x = f^m(g_{i\alpha}(x))$  implies  $x = f^l(x)$ , where  $l = \max\{m-i, i-m\}$  implying  $x \in \{p_{jl\beta}\}$ . By using the previous arguments, one can show that the discontinuity cannot occur unless  $x = p_{jll}$ . This proves that  $\rho_n^1(x)$  cannot be discontinuous unless  $x = p_{jll}$  for some  $l \leq j$ .

Conversely, we show that at every  $x = p_{jll}$ ,  $\rho_n^1(x)$  is discontinuous. This is done in two steps for convenience, namely (a) if  $p_{jll}$  is such that  $l > (n-1)/2$ ,  $C_0(x)$  registers a discontinuity at  $x = p_{jll}$  while (b) if  $l \leq (n-1)/2$ ,  $C_{i\alpha}(x)$  will be discontinuous at  $x = p_{jll}$ .

(a) Consider  $C_0(p_{jll} + \epsilon)$  and writing it as a product of three terms, we have

$$C_0(p_{jll} + \epsilon) = \Theta\left(\epsilon \left[1 - \frac{d}{dx} f^l(x) \Big|_{x=p_{jll}}\right]\right) \prod_{k=1}^{l-1} \Theta(p_{jll} - f^k(p_{jll}) + O(\epsilon)) \prod_{k=1}^{n-l-1} \Theta(p_{jll} - f^{k+l}(p_{jll}) + O(\epsilon)) \quad (10)$$

$f^k(p_{jll}) < p_{jll}$  for all  $k \leq l-1$  and  $n-l-1 < l-1$  if  $l > (n-1)/2$ . This implies that the product terms in the equation (10) are non zero and the first term also can be

made non zero by choosing an  $\epsilon$  of appropriate sign. This proves that if  $l > (n-1)/2$ , at  $x = p_{jl}$ ,  $l \leq n-1$ ,  $j = 1, \dots, N_l$ ,  $C_0(x)$  and hence  $\rho_n^1(x)$  are discontinuous.

(b) We prove that, given  $x = p_{jmm}$ ,  $m \neq i$  and  $m \leq (n-1)/2$ , there exists a  $\Theta$  function in  $C_{i\alpha}(x)$ , see equation (6), with argument  $x - f^l(g_{i\alpha}(x))$ ,  $l \neq i$  such that  $l = i \pm m$  which is discontinuous at  $x = p_{jmm}$ , see for details [18] implying  $C_{i\alpha}(x)$  and hence  $\rho_n^1(x)$  to be discontinuous at  $x = p_{jmm}$ .

Similarly, one can prove that the density corresponding to  $k = n$ , the minimum of the iterates,  $\rho_n^n(x)$  is discontinuous at  $x$  if and only if  $x = p_{jl1}$ , i.e., at the minimum of the periodic orbits.

Finally, it can be proven that at every periodic point  $x = p_{jl\beta}$ , there exists a  $k$  such that the order density  $\rho_n^k(x)$  is discontinuous at  $p_{jl\beta}$ . This result is obtained by subjecting  $\rho_n^k(x)$  to a similar analysis outlined in the case of  $\rho_n^1(x)$ , see [18] for details.

In summary, we have presented a new method to extract all the unstable periodic points of one dimensional chaotic maps. This method uses order statistics. We calculate  $\rho_n^k$  of the logistic map analytically for  $n = 3$  and numerically for  $n > 3$  and illustrate that their discontinuities coincide with the unstable periodic points of all orders less than  $n$ . We demonstrate the applicability of this method to the map constructed out of the successive maxima of the time series  $z(t)$  of Lorenz equations. In both of the above cases topological entropy has been estimated. Further, we give an outline of the proof which establishes this connection between order statistics and unstable periodic orbits for any continuous map with a smooth invariant density.

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